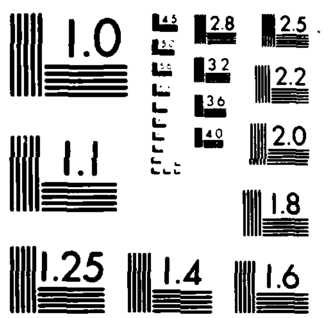
AD-A129 559 AN ITERATED LOGARITHM LAW RESULT FOR EXTREME VALUES FORM GAUSSIAN SEQUENC..(U) NORTH CAROLINA UNIV AT CHAPEL HILL DEPT OF STATISTICS W P MCCORMICK MAR 83 1/1

UNCLASSIFIED TR-29 AFOSR-TR-83-0510 F49620-82-C-0009 F/G 12/1 NL

END
DATE
FILMED
7 83
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

CENTER FOR STOCHASTIC PROCESSES

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

AN ITERATED LOGARITHM LAW RESULT FOR EXTREME VALUES FROM GAUSSIAN SEQUENCES

William P. McCormick

TECHNICAL REPORT #29

March 1983

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 83-0510	2. GOVT ACCESSION NO. AD-A129559	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN ITERATED LOGARITHM LAW RESULT FOR EXTREME VALUES FROM GAUSSIAN SEQUENCES		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
		6. PERFORMING ORG. REPORT NUMBER TR #29
7. AUTHOR(s) William P. McCormick		8. CONTRACT OR GRANT NUMBER(s) F49620-82-C-0009
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics University of North Carolina Chapel Hill NC 27514		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE61102F; 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical & Information Sciences Directorate Air Force Office of Scientific Research Bolling AFB DC 20332		12. REPORT DATE MAR 83
		13. NUMBER OF PAGES 16
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Iterated logarithm; Gaussian sequence; almost sure limit set.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

SEE REVERSE

83 06 20 127

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When D

ITEM #20, CONTINUED:

Let $\{X_n, n\geq 1\}$ be a stationary Gaussian sequence with $EX_1=0$, $EX_1^2=1$ and $r_n=EX_1X_{n+1}$. Let $Z_n^{(i)}$ denote the ith maximum of $X_1,\ldots, X_n$ and $a_n=(\ell n\ell nn)(2\ell nn)^{-1/2}$, $b_n=(2\ell nn)^{1/2} - (\ell n(4\pi\ell nn))/(2(2\ell nn)^{1/2})$. Then assuming $r_n(\ell nn)^2=0(1)$ the set of almost sure limit points of the vectors $((Z_n^{(1)}-b_n)a_n^{-1}, (Z_n^{(2)}-b_n)a_n^{-1},\ldots(Z_n^{(\ell)}-b_n)a_n^{-1})$ is determined. The number of components $\ell=\ell(n)\to\infty$ as $n\to\infty$. This extends a result of Hebbar.

AN ITERATED LOGARITHM LAW RESULT
FOR EXTREME VALUES FROM GAUSSIAN SEQUENCES

William P. McCormick

Abstract

Let $\{X_n, n \geq 1\}$ be a stationary Gaussian sequence with $EX_1=0$, $EX_1^2=1$ and $r_n=EX_1X_{n+1}$. Let $Z_n^{(i)}$ denote the ith maximum of $X_1, \ldots, X_n$ and $a_n=(\ell n \ell n n)(2\ell n n)^{-1/2}$, $b_n=(2\ell n n)^{1/2} - (\ell n(4\pi \ell n n))/(2(2\ell n n)^{1/2})$. Then assuming $r_n(\ell n n)^2=o(1)$ the set of almost sure limit points of the vectors $((Z_n^{(1)}-b_n)a_n^{-1}, (Z_n^{(2)}-b_n)a_n^{-1}, \ldots (Z_n^{(\ell)}-b_n)a_n^{-1})$ is determined. The number of components $\ell=\ell(n)\to\infty$ as $n\to\infty$. This extends a result of Hebbar.

Keywords: Iterated logarithm, Gaussian sequence, almost sure limit set.

This research is supported by AFOSR Grant No. F49620 82 C 0009. A

1. Introduction

Let $\{X_n, n\geq 1\}$ be a stationary Gaussian sequence with $EX_1=0$, $EX_1^2=1$ and $r_n=EX_1X_{n+1}$. Let $Z_n^{(i)}$ denote the ith maximum of $X_1,\ldots, X_n$ that is $Z_n^{(i)}$ equals the n-i+1 order statistic. Set $a_n = \ell n\ell nn/\sqrt{2\ell nn}$ and $b_n = \sqrt{2\ell nn} - \ell n(4\pi\ell nn)/2\sqrt{2\ell nn}$. In [1] Hebbar considers the set of almost sure limit points of the sequence of vectors $\{(\frac{Z_n^{(1)}-b_n}{a_n}, \frac{Z_n^{(2)}-b_n}{a_n},\ldots, \frac{Z_n^{(\ell)}-b_n}{a_n}), n\geq 1\}$. He shows that under the assumption $r_n(\ell nn)^{2+\varepsilon} = O(1)$ for some $\varepsilon>0$ the above sequence has almost sure limit set equal to $\{(x_1,x_2,\ldots, x_\ell): 0\leq x_\ell\leq\ldots\leq x_1$ and $\sum_1^\ell x_i\leq 1\}$. In the present paper we strengthen this result in two directions. We relax the condition on r_n to $r_n(\ell nn)^2=O(1)$ and further we allow the number ℓ of extreme values considered to grow to infinity with n. Let $v_n^{(i)} = \frac{Z_n^{(i)}-b_n}{a_n}$. Then we consider the points in $\mathbb{R}^\infty$ given by $(v_n^{(1)},\ldots, v_n^{(\ell)},0,0\ldots)$ where $\ell=\ell(n)\to\infty$ as $n\to\infty$. In $\mathbb{R}^\infty$ we consider two modes of convergence--pointwise convergence and ℓ_1-convergence. With $\ell(n)$ suitably bounded we show that the almost sure limit set in $\mathbb{R}^\infty$ is given by

$$A = \{(x_1,x_2\ldots): \quad 0\leq x_{i+1}\leq x_i,\ i=1,2,\ldots, \text{ and } \sum_1^\infty x_i\leq 1\}$$

2. Almost sure limit set

We consider two modes of convergence in $\mathbb{R}^\infty$, pointwise which is metrized by $d(\underline{x},\underline{y}) = \sum_{n=1}^\infty (\frac{|x_n-y_n|}{1+|x_n-y_n|})2^{-n}$ and ℓ_1. Let us observe that a point $\underline{x}$ is a limit point of a sequence $\underline{x}_n$ with respect to pointwise convergence if and only if for each fixed ℓ, $(x_1,\ldots, x_\ell)$ is a limit point of $(x_n^{(1)},\ldots, x_n^{(\ell)})$. Therefore with regard to pointwise convergence our extension of Hebbar's result is precisely to weaken the mixing condition on r_n since finite dimensional results suffice to prove this case. Furthermore in this case we consider the almost sure limit points of the sequence $\{(v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(n)}, 0, 0, \ldots), n \geq 1\}$ that is we take $\ell(n) = n$.

However when we consider the random element $(v_n^{(1)}, \ldots, v_n^{(\ell)}, 0, 0, \ldots)$ as a point in ℓ_1 then we must take into account the rate at which $\ell(n)$ grows with n. In this case we prove an iterated logarithm law result with $\ell(n) = [\ell n_3 n]$. In the following we consider the ℓ_1 case only since the pointwise convergence case immediately follows.

The proof closely follows the method in [1] although additional detail is required to accommodate the infinite dimensional setting. However Lemma 6 in [1] receives an entirely different proof here that depends on an extension of a result of Mittal [2].

Remark: Let $\underline{x} = (x_1, x_2, \ldots) \in A$ and assume $x_1 > 0$. Define the following sequences $\lambda_k = [\ell n(\frac{1}{x_1} \ell nk)]$, $s_k = \sum_1^{\lambda_k} x_i$, $s = \sum_1^{\infty} x_i$ (assume s<1) and $\alpha_k = [\exp(k^{\frac{1}{s_k}})]$. Our program will be to show that the sequence $(v_{\alpha_k}^{(1)}, v_{\alpha_k}^{(2)}, \ldots, v_{\alpha_k}^{(\lambda_k)}, 0, 0 \ldots)$, $k \geq 1$ has $\underline{x}$ as a limit point almost surely. Then since $\ell_{\alpha_k} \leq \lambda_k$ and $\ell_{\alpha_k} \to \infty$ as $k \to \infty$ it follows easily that $\underline{x}$ is a limit point of $(v_{\alpha_k}^{(1)}, v_{\alpha_k}^{(2)}, \ldots, v_{\alpha_k}^{(\ell_{\alpha_k})}, 0, 0 \ldots)$. In the lemmas which follow it will be assumed that $r_n(\ell nn)^2 = 0(1)$ and that $s = \sum_1^{\infty} x_i < 1$.

<u>Lemma 1</u>. For any $\varepsilon > 0$ we have

$$(2.1) \quad P\{\sum_{i=1}^{\lambda_k} (v_{\alpha_k}^{(i)} - x_i) > \varepsilon \quad \text{and} \quad v_{\alpha_k}^{(i)} > x_i,\ i = 1, \ldots, \lambda_k,\ \text{i.o.}\} = 0\ .$$

<u>Proof:</u> To establish (2.1) it suffices to prove

$$(2.2) \quad P\{\max_{1 \leq i \leq \lambda_k} (v_{\alpha_k}^{(i)} - x_i) > \varepsilon/\lambda_k,\ \min_{1 \leq i \leq \lambda_k} (v_{\alpha_k}^{(i)} - x_i) > 0,\ \text{i.o.}\} = 0\ .$$

Further by Borel Cantelli to establish (2.2) it suffices to show

$$(2.3)\quad \sum_k [\lambda_k \max_{1\le j\le\lambda_k} P\{v_{\alpha_k}^{(j)} > x_j + \varepsilon/\lambda_k,\ v_{\alpha_k}^{(i)} > x_i,\ 1\le i\le\lambda_k\}] < \infty .$$

Let $\{(y_{\alpha_k}^{(1)}, y_{\alpha_k}^{(2)}, \ldots, y_{\alpha_k}^{(\lambda_k)}),\ k\ge 1\}$ be any triangular array with $y_{\alpha_k}^{(i)} \ge x_i$, $1\le i\le\lambda_k$ and $\max_{1\le i\le\lambda_k} (y_{\alpha_k}^{(i)} - x_i) > \varepsilon/\lambda_k$. Let $\eta_{\alpha_k}^{(i)} = b_{\alpha_k} + y_{\alpha_k}^{(i)} a_{\alpha_k}$. Then we establish (2.3) by showing that

$$(2.4)\quad \sum_{k=1}^{\infty} \lambda_k P\{Z_{\alpha_k}^{(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\} < \infty .$$

Let $Z_n^{*(i)}$ be the ith maximum of a sample of size n of i.i.d. standard normal random variables. Then in order to show (2.4) it suffices to show

$$(2.5)\quad \sum_1^{\infty} \lambda_k P\{Z_{\alpha_k}^{*(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\} < \infty \quad \text{and}$$

$$(2.6)\quad \sum_1^{\infty} \lambda_k \left| P\{Z_{\alpha_k}^{*(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\} - P\{Z_{\alpha_k}^{(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\}\right| < \infty .$$

In considering (2.5) observe that

$$\begin{aligned} &P\{Z_{\alpha_k}^{*(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\} \\ (2.7)\quad &= P\{Z_{\alpha_k}^{*(1)} \ge \eta_{\alpha_k}^{(1)}\} \\ &- \sum_{i=2}^{\lambda_k} P\{Z_{\alpha_k}^{*(1)} > \eta_{\alpha_k}^{(1)}, \ldots, Z_{\alpha_k}^{*(i-1)} > \eta_{\alpha_k}^{(i-1)},\ Z_{\alpha_k}^{*(i)} \le \eta_{\alpha_k}^{(i)}\} \end{aligned}$$

Further it can be easily checked that

$$(2.8)\quad P\{Z_{\alpha_k}^{*(1)} > \eta_{\alpha_k}^{(1)}\} = k^{-\frac{y_{\alpha_k}^{(1)}}{s_k}} + O(\tfrac{1}{\alpha_k}) \quad \text{and}$$

$$(2.9)\quad P\{Z^{*(1)}_{\alpha_k} > \eta^{(1)}_{\alpha_k},\ldots, Z^{*(i-1)}_{\alpha_k} > \eta^{(i-1)}_{\alpha_k}, Z^{*(i)}_{\alpha_k} \le \eta^{(i)}_{\alpha_k}\}$$

$$= k^{-(\frac{1}{s_k}\sum_1^{i-1} y^{(t)}_{\alpha_k})} - k^{-(\frac{1}{s_k}\sum_1^{i} y^{(t)}_{\alpha_k})} + 0(\frac{1}{\alpha_k})$$

Thus by (2.8) and (2.9) we obtain that (2.7) equals $k^{-\frac{1}{s_k}\sum_1^{\lambda_k} y^{(t)}_{\alpha_k}} + 0(\frac{\lambda_k}{\alpha_k})$

Thus since $\sum_1^{\lambda_k} y^{(t)}_{\alpha_k} > s_k + \epsilon\lambda_k^{-1}$ and $s_k \nearrow s = \sum_1^{\infty} x_i \le 1$, (2.5) is established.

Next we consider (2.6). First observe that

$$|P\{Z^{*(i)}_{\alpha_k} \ge \eta^{(i)}_{\alpha_k},\ i=1,\ldots,\lambda_k\} - P\{Z^{(i)}_{\alpha_k} \ge \eta^{(i)}_{\alpha_k},\ i=1,\ldots,\lambda_k\}|$$

$$\le |P\{\eta^{(1)}_{\alpha_k} \le Z^{*(1)}_{\alpha_k} \le z_{\alpha_k}\} - P\{\eta^{(1)}_{\alpha_k} \le Z^{(1)}_{\alpha_k} \le z_{\alpha_k}\}|$$

$$(2.10)\qquad + \sum_{i=2}^{\lambda_k} |P\{\eta^{(1)}_{\alpha_k} \le Z^{*(1)}_{\alpha_k} \le z_{\alpha_k},\ldots, \eta^{(i-1)}_{\alpha_k} \le Z^{*(i-1)}_{\alpha_k} \le z_{\alpha_k}, Z^{*(i)}_{\alpha_k} \le \eta^{(i)}_{\alpha_k}\}$$

$$- P\{\eta^{(1)}_{\alpha_k} \le Z^{(1)}_{\alpha_k} \le z_{\alpha_k},\ldots, \eta^{(i-1)}_{\alpha_k} \le Z^{(i-1)}_{\alpha_k} \le z_{\alpha_k}, Z^{(i)}_{\alpha_k} \le \eta^{(i)}_{\alpha_k}\}|$$

$$+ P\{Z^{*(1)}_{\alpha_k} > z_{\alpha_k}\} + P\{Z^{(1)}_{\alpha_k} > z_{\alpha_k}\}$$

where $z_{\alpha_k} = 2\sqrt{\ell n \alpha_k}$.

It can be checked that

$$(2.11)\qquad \begin{aligned} &P\{Z^{*(1)}_{\alpha_k} > z_{\alpha_k}\} = 0(\frac{1}{\alpha_k}) \quad \text{and} \\ &|P\{Z^{*(1)}_{\alpha_k} > z_{\alpha_k}\} - P\{Z^{(1)}_{\alpha_k} > z_{\alpha_k}\}| \le \alpha_k^{-2(\frac{1-\bar{r}_1}{1+\bar{r}_1})} \end{aligned}$$

where $\bar{r}_x = \sup_{i>x} |r_i|$. Thus by (2.11) $\lambda_k(P\{Z^{*(1)}_{\alpha_k} > z_{\alpha_k}\} + P\{Z^{(1)}_{\alpha_k} > z_{\alpha_k}\})$ is summable on k.

Similarly it is easily checked that

$$(2.12)\quad |P\{Z^{*(1)}_{\alpha_k} \le \eta^{(1)}_{\alpha_k}\} - P\{Z^{(1)}_{\alpha_k} \le \eta^{(1)}_{\alpha_k}\}|\lambda_k \le (\text{CONST.})k^{-(\frac{1+2y^{(1)}_{\alpha_k}}{s_k})}\lambda_k .$$

Since $s_k<1$ and $y^{(1)}_{\alpha_k} \ge x_1 > 0$ for all k, the series in (2.12) is summable.

Now consider a term of the form

$$|P\{\eta^{(1)}_{\alpha_k} \le Z^{*(1)}_{\alpha_k} \le z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} \le Z^{*(i-1)}_{\alpha_k} \le z_{\alpha_k}, Z^{*(i)}_{\alpha_k} \le \eta^{(i)}_{\alpha_k}\}$$

$$(2.13)\quad - P\{\eta^{(1)}_{\alpha_k} \le Z^{(1)}_{\alpha_k} \le z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} \le Z^{(i-1)}_{\alpha_k} \le z_{\alpha_k}, Z^{(i)}_{\alpha_k} \le \eta^{(i)}_{\alpha_k}\}|$$

$$\le S = \sum_{t_1,\dots,t_{i-1}} |P\{\eta^{(1)}_{\alpha_k} \le X^*_{t_1} \le z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} \le X^*_{t_{i-1}} \le z_{\alpha_k}, X^*_t \le \eta^{(i)}_{\alpha_k}$$

for all $t \ne t_1,\dots, t_{i-1}$, $1\le t\le\alpha_k\}|$

where $\{X^*_1, X^*_2,\dots\}$ denotes an i.i.d. sequence of standard normal random variables and where the summation is over all $1 \le t_1,\dots, t_{i-1} \le \alpha_k$ and $t_u \ne t_v$ of $u \ne v$.

Let $0<\theta<1$ be fixed and to be specified later. We write

$$S = S_0 + S_1 + \dots + S_{i-2}$$

where S_u denotes the sum over all $t_1,\dots, t_{i-1}$ such that when the t's are ordered $t_{(1)} <\dots< t_{(i-1)}$ there are exactly u indices h where $t_{(h+1)} - t_{(h)} < \alpha_k^\theta$.

Consider S_0. We have

$$|P\{\eta^{(1)}_{\alpha_k} < X^*_{t_1} < z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} < X^*_{t_{i-1}} < z_{\alpha_k}, X^*_t \le \eta^{(i)}_{\alpha_k},\ t\ne t_1,\dots, t_{i-1} \text{ and } 1\le t\le\alpha_k\}$$

$$- P\{\eta^{(1)}_{\alpha_k} < X_{t_1} < z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} < X_{t_{i-1}} < z_{\alpha_k}, X_t \le \eta^{(i)}_{\alpha_k}, t\ne t_1,\dots, t_{i-1} \text{ and } 1\le t\le\alpha_k\}|$$

$$\le (\text{Const.})(T_0 + \sum_{0\le u\ne v\le i-1} T_{u,v})$$

where $T_0 = \sum^{(0)}_{s,t} |r|\phi(\eta^{(i)}_{\alpha_k}, \eta^{(i)}_{\alpha_k}, |r|)$

$$\cdot\ P\{\eta^{(1)}_{\alpha_k} < X_{t_1} < z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} < X_{t_{i-1}} < z_{\alpha_k} | X_s = \eta^{(i)}_{\alpha_k}, X_t = \eta^{(i)}_{\alpha_k}\}$$

where $r = r_{s,t}$ and $\sum^{(0)}$ is summation over all $s \neq t$ and $s,t \neq t_1,\ldots, t_{i-1}$, $1 \le s,t \le \alpha_k$ and where $\phi(\cdot,\cdot,r)$ denotes the bivariate normal density with zero means, unit variances and correlation r. Further for $v>0$,

$$T_{0,v} = \sum_s^{(0,v)} |r|\phi(\eta_{\alpha_k}^{(i)}, \eta_{\alpha_k}^{(v)}, |r|)$$

$$\cdot P\{\eta_{\alpha_k}^{(1)} < X_{t_1} < z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(v-1)} < X_{t_{v-1}} < z_{\alpha_k}, \eta_{\alpha_k}^{(v+1)} < X_{t_{v+1}} < z_{\alpha_k},$$

$$\ldots \eta_{\alpha_k}^{(i-1)} < X_{t_{i-1}} < z_{\alpha_k} | X_{t_v} = \eta_{\alpha_k}^{(v)}, X_s = \eta_{\alpha_k}^{(i)}\}$$

where the sum is over $s \neq t_1, \ldots, t_{i-1}$ and $1 \le s \le \alpha_k$ and $r = r_{st_v}$.

$T_{u,0}$ is defined in exactly the same way and finally for $u,v>0$

$$T_{u,v} = |r|\phi(\eta_{\alpha_k}^{(u)}, \eta_{\alpha_k}^{(v)}, |r|)$$

$$P\{\eta_{\alpha_k}^{(1)} < X_{t_1} < z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(u-1)} < X_{t_{u-1}} < z_{\alpha_k}, \eta_{\alpha_k}^{(u+1)} < X_{t_{u+1}} < z_{\alpha_k},$$

$$\ldots \eta_{\alpha_k}^{(v-1)} < X_{t_{v-1}} < z_{\alpha_k}, \eta_{\alpha_k}^{(v+1)} < X_{t_{v+1}} < z_{\alpha_k}, \ldots,$$

$$\eta_{\alpha_k}^{(i-1)} \le X_{t_{i-1}} \le z_{\alpha_k} | X_{t_u} = \eta_{\alpha_k}^{(u)}, X_{t_v} = \eta_{\alpha_k}^{(v)}\}.$$

We will give details only for the sum T_0 since the other sums are handled in the same way. For T_0 first consider the case when

$$(2.14) \quad \min\{|s-t_u|, |t-t_u|, u=1,\ldots, i-1\} > \alpha_k^{\theta}$$

In evaluating T_0 we need to evaluate

$$(2.15) \quad P\{\eta_{\alpha_k}^{(1)} \le X_{t_1} \le z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} \le X_{t_{i-1}} \le z_{\alpha_k} | X_s = \eta_{\alpha_k}^{(i)}, X_t = \eta_{\alpha_k}^{(i)}\}$$

when

(2.16) $t_{(h+1)} - t_{(h)} > \alpha_k^{\theta}$, $h=1,\ldots, i-2$

and (2.14) hold. Now subject to (2.14) we have that

$$(2.17)\quad E(X_{t_i} | X_s = \eta_{\alpha_k}^{(i)},\ X_t = \eta_{\alpha_k}^{(i)}) = 0((\ell n\alpha_k)^{-3/2}) \quad \text{and}$$

$$\mathrm{CORR}(X_{t_u}, X_{t_v} | X_s, X_t) = r_{t_u,t_v} + 0((\ell n\alpha_k)^{-4})$$

Therefore by (2.17) the probability in (2.15) is at most

$$(2.18)\quad P\{\eta_{\alpha_k}^{(u)} - c(\ell_n\alpha_k)^{-3/2} \le X_{t_u} \le z_{\alpha_k} + c(\ell n\alpha_k)^{-3/2},\ u=1,\ldots, i-1\}$$

for some constant c not depending on k. Conditioning on X_{t_1} yields that (2.18) is at most

$$(1-\Phi(b_{\alpha_k}(1-c\bar{r})))P\{b_{\alpha_k}(1-c\bar{r})^2 \le X_{t_u} \le z_{\alpha_k}(1-c\bar{r})^2,\ u=2,\ldots, i-1\}$$

where $\bar{r} = \bar{r}_{\alpha_k^{\theta}}$.

Iterating the procedure yields that (2.18) is at most

$$(2.19)\quad \prod_{u=1}^{i-1} [1-\Phi(b_{\alpha_k}(1-c\bar{r})^u]$$

Finally since $i \le \lambda_k = [\ell n(\frac{\ell nk}{x_1})]$ and $(1-c\bar{r})^u \ge 1 - 2\lambda_k c\bar{r}$, (2.19) is at most

$$(2.20)\quad [1-\Phi(b_{\alpha_k} - c\frac{\ell nk}{(\ell n\alpha_k)^{3/2}})]^{(i-1)} . \qquad c \text{ is some constant.}$$

In the same way it can be checked that if for some u_0, $|s-t_{u_0}| < \alpha_k^{\theta}$ but $|t-t_u| > \alpha_k^{\theta}$, $u=1,\ldots, i-1$ or the same case with s and t interchanged then (2.15) is at most

$$(2.21)\quad (1-\Phi(\gamma b_{\alpha_k}))[1-\Phi(b_{\alpha_k} - c\ \frac{\ell nk}{(\ell n\alpha_k)^{3/2}})]^{(i-2)}$$

where $\gamma > 0$.

And finally if both $|s-t_{u_0}| < \alpha_k^{\theta}$ and $|t-t_{v_0}| < \alpha_k^{\theta}$ for some u_0 and v_0 then (2.15) is at most

(2.22) $$(1-\Phi(\gamma b_{\alpha_k}))^2[1-\Phi(b_{\alpha_k} - c\frac{\ell nk}{(\ell n\alpha_k)^{3/2}})]^{(i-3)}$$

Thus we have that provided (2.16) holds

(2.23) $$T_0 \le \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(2y_{\alpha_k}^{(i)}+1)}}[1-\Phi(b_{\alpha_k} - c\frac{\ell nk}{(\ell n\alpha_k)^{3/2}})]^{(i-1)} = \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(2y_{\alpha_k}^{(i)}+1)}}\frac{1}{\alpha_k^{(i-1)}}$$

Similarly if (2.16) holds we find

(i) $$T_{0,v} \le \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y_{\alpha_k}^{(i)}+y_{\alpha_k}^{(v)}+1)}}\frac{1}{\alpha_k^{(i-1)}}$$

(2.24) (ii) $$T_{u,0} \le \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y_{\alpha_k}^{(i)}+y_{\alpha_k}^{(u)}+1)}}\frac{1}{\alpha_k^{(i-1)}}$$

(iii) $$T_{u,v} \le \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y_{\alpha_k}^{(u)}+y_{\alpha_k}^{(v)}+1)}}\frac{1}{\alpha_k^{(i-1)}}$$

Thus by (2.23) and (2.24)

(2.25) $$S_0 \le [\frac{(\text{CONST.})}{\alpha_k^{(i-1)}}\frac{(\ell n_2 k)^2}{k^{(1+2x_i)/s_k}}](\alpha_k^{(i-1)}) \le \frac{1}{k^{1+e}}$$

for some $e > 0$.

Next consider S_h. For simplicity let us consider a summand in (2.13) when $t_1<t_2<\ldots< t_{i-1}$ and

(2.26) $0 < t_2 - t_1,\ t_3 - t_2, \ldots, t_{n+1} - t_n \le \alpha_k^\theta$ and $t_{u+1} - t_u > \alpha_k^\theta$ $u=h+1, \ldots, i-1$.

Then

$$|P\{\eta_{\alpha_k}^{(1)} \le X_{t_1}^* \le z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} \le X_{t_{i-1}}^* \le z_\alpha,\ X_t \le \eta_{\alpha_k}^{(i)}$$

for all $t \ne t_1, \ldots, t_{i-1},\ 1 \le t \le \alpha_k\}$

$$- P\{\eta_{\alpha_k}^{(1)} \le X_{t_1} \le z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} \le X_{t_{i-1}} \le z_\alpha,\ X_t \le \eta_{\alpha_k}^{(i)}$$

for all $t \ne t_1, \ldots, t_{i-1},\ 1 \le t \le \alpha_k\}|$

$$\le T_0 + \sum_{0 \le u \ne v \le i-1} T_{u,v}$$

where the $T_{u,v}$ have the same meaning as before except now condition (2.26) holds. Consider T_0. We need to evaluate

$$P\{\eta_{\alpha_k}^{(1)} < X_{t_1} < z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} < X_{t_{i-1}} < z_{\alpha_k} | X_s = \eta_{\alpha_k}^{(i)} X_t = \eta_{\alpha_k}^{(i)}\}$$

Suppose that (2.14) holds. Then the above conditional probability is at most the expression given at (2.18). Therefore let us consider

(2.27) $P\{\eta_{\alpha_k}^{(u)} \le X_{t_u} \le z_{\alpha_k},\ u=1, \ldots, i-1\}$

subject to condition (2.26).

Let $K = K(\alpha_k) = [\exp(\sqrt{\ell n \alpha_k})]$. Suppose that in addition to (2.26) we have that

(2.28) $t_2 - t_1, \ldots, t_{m+1} - t_m \le K$, and $K < t_{m+2} - t_{m+1}, \ldots, t_{i-1} - t_{i-2}$.

Then given (2.26) and (2.28), (2.27) equals at most

$$\prod_{u=0}^{m} (1-\Phi(b_{\alpha_k}(1- c\bar{r}_1)^u)) \cdot \prod_{u=m+1}^{h} (1-\Phi(b_{\alpha_k}(1-c\bar{r}_K)^u)) \cdot \prod_{u=h+1}^{i=2} (1-\Phi(b_{\alpha_k}(1-c\bar{r}_{\alpha_k^\theta})^u))$$

$$\le (\text{CONST.})\alpha_k^{-Q}$$

where $Q = 1 + \delta_1 - \delta_1^{m+1} + (h-m)\delta_K^{m+1} + (i-2-h)$ and $\delta_1 = 1 - c\bar{r}_1$, $\delta_K = 1 - c\bar{r}_K$ where c is some constant and without loss of generality we can assume $c\bar{r}_1 < 1$ because if necessary we can work with the sequence $\{X_{mn}, n\geq 1\}$ where m is some fixed integer.

Then as before we find that

$$\text{(i)} \quad T_0 \leq \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(2y_{\alpha_k}^{(i)}+1)}} \frac{1}{\alpha_k^Q}$$

(2.29) $$\text{(ii)} \quad T_{u,0} \leq \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y_{\alpha_k}^{(u)}+y_{\alpha_k}^{(i)}+1)}} \frac{1}{\alpha_k^Q}$$

and

$$\text{(iii)} \quad T_{u,v} \leq \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y_{\alpha_k}^{(u)}+y_{\alpha_k}^{(v)}+1)}} \frac{1}{\alpha_k^Q}$$

Thus by the inequalities in (2.29) we have that

$$S_h \leq \sum_{m=0}^{h} \left(\frac{(\text{CONST.})}{\alpha_k^Q}\right)\left(\frac{(\ell n_2 k)^2}{k^{\frac{1+2x_i}{s_k}}}\right)\left(\alpha_k^{[i-h-1+\theta(h-m)]}K^m\right)$$

from which it can be easily checked that

(2.30) $S_n \leq \alpha_k^{-f}$ for some $f > 0$ not depending on h.

Finally from (2.25) and (2.30) we have that

$$S \leq \frac{(\text{CONST.})}{k^{1+e}} \quad k \geq 1 \text{ for some } e > 0 .$$

Hence (2.6) holds completing the proof of lemma 1.

Lemma 2. Let $c_n^{(i)} = b_n + x_i a_n$. Then

(2.31) $P\{Z_{\alpha_k}^{(i)} > c_{\alpha_k}^{(i)},\ i=1,\dots,\lambda_k)\ \text{i.o.}\} = 1$.

Proof: Let $\beta_k = [1/2\ \alpha_k]$ and let $\tilde{Z}\alpha_k^{(i)}$ be the ith maximum of the random variables $X_{\alpha_k-\beta_k+1},\dots,X_{\alpha_k}$. Let z_{α_k} be as in Lemma 1 and define $I_k = \prod_{i=1}^{\lambda_k} I_{[c_{\alpha_k}^{(i)}<\tilde{Z}_{\alpha_k}^{(i)}<z_{\alpha_k}]}$.

Let $J_n = \sum_{k=[n^a]}^{n} I_k$ where $0<a<1$ is a fixed real number. Then to show (2.31) it suffices to show

(2.32)
(i) $EJ_n \to \infty$ as $n \to \infty$ and

(ii) $J_n/EJ_n \xrightarrow{P} 1$ as $n \to \infty$.

The proof of (2.32) follows the method of proof of Lemma 3 in [1] with changes similar to those in our Lemma 1. Therefore the details of this proof will be omitted.

Remark: The sequence $(v_{\alpha_k}^{(1)},\dots,v_{\alpha_k}^{(\lambda_k)},0,0\dots)$ has $\underline{x}$ as an almost sure limit point. To see this let $N_n = \{\omega:\ \sum_1^{\lambda_k}(v_{\alpha_k}^{(i)}-x_i) > 1/n,\ v_{\alpha_k}^{(i)} > x_i,\ i=1,\dots,\lambda_k\ \text{i.o.}\}$ and $N = \bigcup_{n=1}^{\infty} N_n$. Let $A = \{\omega:\ v_{\alpha_k}^{(i)} > x_i,\ i=1,\dots,\lambda_k\ \text{i.o.}\}$ and $\Lambda = A \cap N^c$. It is easy to check that if $\omega\in\Lambda$, then $(v_{\alpha_k}^{(1)}(\omega),\dots,v_{\alpha_k}^{(\lambda_k)}(\omega),0,0\dots)$ has $\underline{x}$ as a limit point and by Lemmas 1 and 2 $P(\Lambda) = 1$. Therefore by the Remark preceding Lemma 1, we have that $\underline{x}$ is an almost sure limit point of $(v_{\alpha_k}^{(1)},\dots,v_{\alpha_k}^{(\ell\alpha_k)},\ 0,0,\dots)$.

Lemma 3. Let $\underline{x} = (x_1,x_2,\dots)$ be any point in $\mathbb{R}^{\infty}$ with $0\le x_{i+1}\le x_i$, $i=1,2,\dots$ and $\sum_1^{\infty} x_i>1$. Then $\underline{x}$ cannot be an a.s. limit point of the sequence $(v_n^{(1)},\dots,v_n^{(\ell_n)},0,0\dots)$

Proof: Let m be such that $s_m = \sum_1^m x_i > 1$, and $x_m > 0$. Let $z_i = (\frac{s_m+1}{2s_m})x_i$, $i=1,\ldots, m$. Then since $\sum_1^m z_i > 1$, it follows as in [1] that $P\{v_n^{(i)} > z_i,\ i=1,\ldots, m \text{ i.o.}\} = 0$. Let $N = \{\omega: v_n^{(1)} > z_1,\ldots, v_n^{(m)} > z_m, \text{ i.o.}\}$. Then if $\omega \in N^c$, $\underline{x}$ cannot be a limit point of $(v_n^{(1)},\ldots, v_n^{(\ell n)},0,0,\ldots)$ because for all n sufficiently large

$$\sum_1^{\ell(n)} |v_n^{(i)}-x_i| \geq \min_{1\leq i\leq m} (x_i-z_i) = (\frac{s_m+1}{2s_m})x_m .$$

A useful uniform bound on the tail probabilities of the normalized maxima for a Gaussian sequence is provided by Lemma 1 in [2]. We state a version of this result which is suited to our problem.

Lemma 4. Let $c_n = \sqrt{2\ell nn}$. Let $\{X_{k,n}\}$, $k=1,\ldots,n$, $n=1,2,\ldots$ be a triangular array of standard normal random variables. Then setting $r_n(i,j) = EX_{i,n}X_{j,n}$, $M_n = \max_{1\leq k\leq n} X_{k,n}$ and $\delta_n(x) = \sup_{|i-j|\geq x} |r_n(i,j)|$ we have

$$e^{tA^2} P\{c_n(M_n-b_n) \leq -A\} = o(1) \quad \text{as } A \to \infty$$

uniformly in n for all t in a neighborhood of zero provided

(i) $\overline{\lim_{n\to\infty}}\ \delta_n(1) < 1$

(ii) $\delta_n(n^\alpha)\ell nn = 0(1)$ for some fixed $0<\alpha<1$.

Lemma 5. For any fixed positive integer ℓ and $\varepsilon > 0$, $P\{Z_n^{(\ell)} < b_n - \varepsilon a_n, \text{ i.o.}\} = 0$. It is easily checked that it is sufficient to show $P\{Z_{n_k}^{(\ell)} < b_{n_{k+1}} - \varepsilon a_{n_{k+1}}, \text{ i.o.}\}=0$.

Also since for k sufficiently large $b_{n_{k+1}} - \varepsilon a_{n_{k+1}} < b_{n_k} - \varepsilon/2\ a_{n_k}$ it is enough to show

(2.33) $P\{Z_{n_k}^{(\ell)} < b_{n_k} - \varepsilon\ a_{n_k}, \text{ i.0.}\} = 0$.

Observe that

$$P\{Z_n^{(\ell)} < b_n - \varepsilon\, a_n\} \le P\{Z_n^{(1)} > b_n + 2a_n\}$$

(2.34)

$$+ \sum_{i=2}^{\ell} P\{Z_n^{(i)} < b_n - \varepsilon\, a_n < Z_n^{(i-1)} < Z_n^{(1)} < b_n + 2a_n\}$$

Now

$$\text{(2.35)} \quad P\{Z_n^{(1)} > b_n + 2a_n\} \le n(1-\Phi(b_n + 2a_n)) = \frac{1}{(\ell nn)^2}$$

Further we have that

$$P\{Z_n^{(i)} < b_n - \varepsilon\, a_n < Z_n^{(i-1)} < Z_n^{(1)} < b_n + 2a_n\}$$

(2.36)

$$= \sum_{t_1,\ldots,t_{i-1}} P\{X_j \le b_n - \varepsilon\, a_n,\ j \ne t_1,\ldots,\ t_{i-1},\ 1 \le j \le n$$

and $\quad b_n - \varepsilon\, a_n < X_{t_u} < b_n + 2a_n,\ u=1,\ldots,\ i-1\}$

For a fixed $0<\theta<1$ and n sufficiently large

$$P\{X_j \le b_n - \varepsilon\, a_n,\ j \ne t_1,\ldots,\ t_{i-1},\ 1 \le j \le n$$

and

$$b_n - \varepsilon\, a_n < X_{t_u} < b_n + 2a_n,\ u=1,\ldots,\ i-1\}$$

$$\le \int_{b_n-\varepsilon a_n}^{b_n+2a_n} \cdots \int_{b_n-\varepsilon a_n}^{b_n+2a_n} P\{X_j \le b_n - \varepsilon\, a_n,\ |j-t_u| > n^{\theta},$$

(2.37) $\quad u = 1,\ldots,\ i-1,\ 1 \le j \le n \,|\, X_{t_u} = x_u,\ u = 1,\ldots,\ i-1\}$

$$dP\{X_{t_1} \varepsilon dx_1,\ldots,\ X_{t_{i-1}} \varepsilon dx_{i-1}\}$$

$$\le P\{\tilde{X}_j \le b_n - \varepsilon/2\ a_n,\ |j-t_u| > n^{\theta},\ u=1,\ldots,\ i-1,\ 1 \le j \le n\}$$

$$P\{b_n - \varepsilon\, a_n \le X_{t_u} \le b_n + 2a_n,\ u=1,\ldots,\ i-1\}$$

where $\mathrm{CORR}(\tilde{X}_j, \tilde{X}_k) = r_{jk} + O(\bar{r}^{-2}_{n^\theta})$.

Let $1 \le t_{1,n}, \ldots, t_{i-1,n} \le n$ be chosen to maximize

$$P\{\tilde{X}_j \le b_n - \varepsilon/2\ a_n,\ |j-t_u| > n^\theta,\ u=1,\ldots,\ i-1,\ 1 \le j \le n\}$$

Let $Y_{1,m} Y_{2,m} \ldots Y_{m,m}$ represent the $\tilde{X}_j$, $|j-t_u| > n^\theta$, $u=1,\ldots,\ i-1$, $1 \le j \le n$ in their natural order and let $M_m = \max_{1 \le k \le m} Y_k$. Note $n - 2(i-1)n^\theta \le m \le n$. Then the sum in (2.36) is at most

$$(2.38)\quad P\{M_m \le b_m - \varepsilon/4\ a_m\} \cdot \sum_{t_1,\ldots,t_{i-1}} P\{b_n - \varepsilon\ a_n \le X_{t_u} \le b_n + 2a_n,\ u=1,\ldots,i-1\}$$

It is easily checked that the $Y_{k,m}$ satisfy the hypothesis of Lemma 4. Hence

$$(2.39)\quad P\{M_m \le b_m - \varepsilon/4\ a_m\} \le e^{-c(\ell n \ell n n)^2}$$

for some constant $c > 0$ not depending on n.

Further if $\min\{|t_u - t_v|:\ 1 \le u < v \le i-1\} \ge n^\theta$, then following the approach in Lemma 1 one can check that

$$P\{b_n - \varepsilon\ a_n \le X_{t_u} \le b_n + 2a_n,\ u=1,\ldots,\ i-1\} \le (1-\Phi(b_n - i\varepsilon a_n))^{i-1}$$

(2.40)

$$= (1/n\ e^{i\varepsilon \ell n \ell n n})^{i-1}$$

While if there are exactly h indices say $u_1, \ldots, u_h$ such that when the t's are ordered $t_{(u_1+1)} - t_{(u_1)} < n^\theta, \ldots, t_{(u_h+1)} - t_{(u_h)} < n^\theta$ then

$$P\{b_n - \varepsilon\ a_n \le X_{t_u} \le b_n + 2a_n,\ u=1,\ldots,\ i-1\}$$

$$\le (1-\Phi(b_n\delta))^h (1-\Phi(b_n - i\varepsilon a_n))^{i-h-1}$$

where $0<\delta<1$ is some constant not depending on n

$$(2.41)\quad \le (1/n)^{h\delta^2 + i-h-1}\ e^{\varepsilon i^2 (\ell n \ell n n)}$$

Therefore by choosing $\theta < \delta^2$ we have by (2.39), (2.40) and (2.41) that (2.38) is at most $e^{-c(\ell n \ell n n)^2}$ for some $c > 0$. Therefore by (2.34), (2.35) and the above we see that

$$P\{Z_n^{(\ell)} < b_n - \varepsilon\, a_n\} \le \frac{1}{(\ell n n)^2} \; .$$

Since this series evaluated at the n_k is summable on k, (2.33) holds completing the proof of Lemma 5.

Theorem 1. Under the assumptions of Lemma 1 the almost sure limit points of the sequence $(v_n^{(1)},\dots, v_n^{(\ell n)},0,0,\dots)$ in ℓ_1 coincide with the set

$$A = \{(x_1,x_2,\dots):\ \ 0 \le x_{i+1} \le x_i,\ i=1,2,\dots,\ \sum_1^{\infty} x_i \le 1\}$$

Proof: Lemmas 1 and 2 establish that each point of A is an almost sure limit point while Lemmas 3 and 5 establish that no point in A^c can be an almost sure limit point.

REFERENCES

[1] Hebbar, V.: A law of the iterated logarithm for extreme values from Gaussian sequences. Z. Wahrsch. verw. Geb. 48, 1-16 (1979).

[2] Mittal, Y.: Limiting behavior of maxima in stationary Gaussian sequences. Ann. Prob. 2, 231-242 (1974).

D

FI